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On global existence of strong and classical solutions of Navier-Stokes equations

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Abstract This short note studies the problem of a global expansion of local results on existence of strong and classical solutions of Navier-Stokes equations in $\mathbb{R}^3$.

Keywords Navier-Stokes-Equations · Strong Solutions · Dilation Symetry

1 Introduction

In this short note we exploit a very simple idea of global expansion of regularity by means of dilation symmetry, which is well known for systems in $\mathbb{R}^n$.

Let a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ be symmetric with respect to a dilation of the argument, i.e. $\exists \alpha \in \mathbb{R}$ such that

$$f(e^s u) = e^{(\alpha+1)s} f(u), \quad \forall u \in \mathbb{R}^n, \forall s, \in \mathbb{R}$$

If $u(\cdot) : [0, +\infty) \to \mathbb{R}^n$ is a classical solution of

$$\frac{du}{dt} = f(u), \quad t > 0$$

with the initial condition $u(0) = u_0$ then

$$u_s(t) := e^s u(e^{\alpha s} t)$$

is defined on $[0, +\infty)$ and, due to symmetry, we derive

$$\frac{du_s}{dt} = e^{(\alpha+1)s} f(u(e^{\alpha s} t)) = f(u_s(t)), \quad t > 0,$$

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i.e. u_s is a classical solution of the same differential equation with the initial condition $u_s(0) = e^s x_0$, where $s \in \mathbb{R}$.

Let $\exists \varepsilon > 0$ such that a classical solution of the differential equation exists on $[0, +\infty)$ for any initial value $u(0) = u_0 \in B_\varepsilon := \{u \in \mathbb{R}^n : |u| < \varepsilon\}$. To construct a solution for $u_0 \notin B_\varepsilon$ we first need to scale $u_0 \to e^{s_0} u_0$, where $s_0 \in \mathbb{R}$ is such that

$$|e^{s_0} u_0| < \varepsilon.$$

If $u(, e^{s_0} u_0)$ is a solution with the initial condition $u(0) = \lambda_0 u_0$ then $\tilde{u}(t) = e^{-s_0} u(e^{-\alpha s_0}, e^{s_0} u_0)$ is also a solution of the considered system and, obviously,

$$\tilde{u}(0) = e^{s_0} u(0, e^{s_0} u_0) = e^{-s_0} e^{s_0} u_0 = u_0.$$

In this paper we use the dilation symmetry (see, [1, formula (1.5)]) of the Navier-Stokes equations in $\mathbb{R}^3$

$$\begin{aligned} \partial_t u &= \nu \Delta u - (u \cdot \nabla) u - \nabla p, \\ \mathbf{0} &= \mathrm{div} u \end{aligned}$$

where u denotes the velocity of a fluid, p denotes the scalar pressure and $\nu > 0$ denotes viscosity of the fluid, in order to understand when a global-in-time existence of strong or classical solutions of the Navier-Stokes equations for *small initial data* implies the existence of global-in-time strong or classical solutions for *large initial data*. We refer the reader, for example, to [2] and [4], for more details about global-in-time existence of strong solutions for small initial data.

Mainly, the standard *notation* is utilized through the paper, e.g. $\mathbb{R}$ is the field of real numbers; $L^1_{loc}((0,T)\times\mathbb{R}^n, \mathbb{R})$ denotes the space of locally integrable functions $(0,T) \times \mathbb{R}^n \to \mathbb{R}$; $L^p(\mathbb{R}^n, \mathbb{R}^m)$, $1 \leq p \leq +\infty$ is a Lebesgue space of function $\mathbb{R}^n \to \mathbb{R}^m$ with the norm $\|\cdot\|_p$; $C^\infty_c((0,T) \times \mathbb{R}^n, \mathbb{R}^m)$ is a space of smooth functions $(0,T) \times \mathbb{R}^n \to \mathbb{R}^m$ with compact support and $C^\infty_0([0,T) \times \mathbb{R}^n, \mathbb{R}^m)$ is a space of smooth functions which vanish at infinity, where $0 < T \leq \infty$. For composition of operators A, B we also use the notation $A \circ B$.

Let $L^p_\mu(\mathbb{R}^n, \mathbb{R}^m)$ denotes the following normed vector space of functions $\mathbb{R}^n \to \mathbb{R}^m$

$$L^p_\mu(\mathbb{R}^n, \mathbb{R}^m) := \{u : \|u\|_{p,\mu} < +\infty\}, \quad \mu \in \mathbb{R}$$

$$\|u\|_{p,\mu} := \left(\int_{\mathbb{R}^n} |x|^{\mu p} |u(x)|^p \, dx \right)^{1/p}, \quad 0 < p < \infty$$

$$\|u\|_{\infty,\mu} := \mathrm{ess\,sup}(|x|^\mu u(x)), \quad p = \infty,$$

which can be treated as a wighted L_p.

2 Preliminaries: Dilations in functional spaces

The following lemmas deal with the most common dilation groups in functional spaces.

Lemma 1 *The operator* $\mathbf{d}(s)$ *given by*

$$(\mathbf{d}(s)z)(x) = e^{\alpha s} z(e^{\gamma s}t, e^{\beta s}x), \tag{1}$$

where $s \in \mathbb{R}$*,* z *is a function* $(0,T) \times \mathbb{R}^n \to \mathbb{R}^m$*,* $0 < T \leq +\infty$*,* $x \in \mathbb{R}^n$ *and* $\alpha, \beta, \gamma \in \mathbb{R}$ *are constant parameters,*

- *maps* $C_c^\infty((0,T) \times \mathbb{R}^n, \mathbb{R}^m)$ *onto* $C_c^\infty((0, e^{-\gamma s}T) \times \mathbb{R}^n, \mathbb{R}^m)$*;*
- *maps* $C_0^\infty([0,T) \times \mathbb{R}^n, \mathbb{R}^m)$ *onto* $C_0^\infty([0, e^{-\gamma s}T) \times \mathbb{R}^n, \mathbb{R}^m)$*.*

The inverse operator is given by $[\mathbf{d}(s)]^{-1} = \mathbf{d}(-s)$*.*

Proof 1) Since the linear function $(t,x) \to (e^{\gamma s}t, e^{\beta s}x)$ maps a compact in $(0,T) \times \mathbb{R}^n$ to a compact in $(0, e^{-\gamma s}T) \times \mathbb{R}^n$ then $\mathbf{d}(s)$ defined on whole $C_c^\infty((0,T), \mathbb{R}^n)$ and if $z \in C_c^\infty((0,T), \mathbb{R}^n)$ (i.e. z is smooth and has a compact support in $(0,T) \times \mathbb{R}^n$) then $\mathbf{d}(s)z \in C_c^\infty((0, e^{-\gamma s}T), \mathbb{R}^n)$ (i.e. $\mathbf{d}(s)z$ is also smooth, but it has a compact support in $(0, e^{-\gamma s}T) \times \mathbb{R}^n$). Obviously, $(\mathbf{d}(s) \circ \mathbf{d}(-s))z = (\mathbf{d}(-s) \circ \mathbf{d}(s))z = z$ for any $z \in C_c^\infty((0,T), \mathbb{R}^n)$ and any $s \in \mathbb{R}$.

Let us show that $\mathbf{d}(s)$ maps $C_c^\infty((0,T), \mathbb{R}^n)$ onto $C_c^\infty((0, e^{-\gamma s}), \mathbb{R}^n)$. Suppose the opposite: $\exists z^* \in C_c^\infty((0, e^{-\gamma s}T), \mathbb{R}^n)$ such that $z^* \neq \mathbf{d}(s)y$, $\forall y \in C_c^\infty((0,T), \mathbb{R}^n)$. This is impossible, since $\mathbf{d}(-s)z^* \in C_c^\infty((0,T), \mathbb{R}^n)$ and $z^* = (\mathbf{d}(s) \circ \mathbf{d}(-s))z^* \in \mathbf{d}(s)C_c^\infty((0,T), \mathbb{R}^n)$.

2) The proof for C_0^∞ is almost identical. Obviously, if $z \in C_0^\infty([0,T), \mathbb{R}^n)$ (i.e. z is smooth and vanishing at infinity) then $\mathbf{d}(s)z \in C_0^\infty([0, e^{-\gamma s}T), \mathbb{R}^n)$ (i.e. $\mathbf{d}(s)z$ is also smooth and vanishing at infinity) and $\mathbf{d}(s) \circ \mathbf{d}(-s)z = \mathbf{d}(-s) \circ \mathbf{d}(s)z = z$ for any $z \in C_0^\infty([0,T), \mathbb{R}^n)$.

Let us show that $\mathbf{d}(s)$ maps $C_0^\infty([0,T), \mathbb{R}^n)$ onto $C_0^\infty([0, e^{-\gamma s}T), \mathbb{R}^n)$. Suppose the opposite: there exists $z^* \in C_0^\infty([0, e^{-\gamma s}T), \mathbb{R}^n)$ such that $\mathbf{d}(s)y \neq z^*$, $\forall y \in C_0^\infty([0,T), \mathbb{R}^n)$. This is impossible since $\mathbf{d}(-s)z^* \in C_0^\infty([0,T), \mathbb{R}^n)$ and $z^* = \mathbf{d}(s) \circ \mathbf{d}(-s)z^* \in \mathbf{d}(s)C_0^\infty([0,T), \mathbb{R}^n)$.

Lemma 2 *The operator* $\mathbf{d}(s)$ *given by*

$$(\mathbf{d}(s)z)(x) = e^{\alpha s} z(e^{\beta s}x), \tag{2}$$

where $s \in \mathbb{R}$*,* z *is a function* $\mathbb{R}^n \to \mathbb{R}^m$*,* $x \in \mathbb{R}^n$ *and* $\alpha, \beta \in \mathbb{R}$ *are constant parameters, is*

– *a linear bounded invertible operator on* $L^p(\mathbb{R}^n,\mathbb{R}^m)$,

$$\|\mathbf{d}(s)z\|_p = e^{(\alpha-n\beta/p)s}\|z\|_p, \quad z\in L^p(\mathbb{R}^n,\mathbb{R}^m), s\in\mathbb{R},$$

– *a linear bounded invertible operator on* $L^p_\mu(\mathbb{R}^n,\mathbb{R}^m)$,

$$\|\mathbf{d}(s)z\|_p = e^{(\alpha-\beta(\mu+n/p))s}\|z\|_p, \quad z\in L^p(\mathbb{R}^n,\mathbb{R}^m), s\in\mathbb{R},$$

where $0<p\leq\infty$. *The inverse operator is given by* $[\mathbf{d}(s)]^{-1}=\mathbf{d}(-s)$.

Proof Notice that $L^p=L^p_0$.

Let $1\leq p<\infty$. If $z\in L^p_\mu(\mathbb{R}^n,\mathbb{R}^m)$ then

$$\int_{\mathbb{R}^n}|x|^{\mu p}|z(x)|^p\,dx<+\infty$$

and

$$\int_{\mathbb{R}^n}|x|^{\mu p}|z(x)|^p\,dx = e^{n\beta s}\int_{\mathbb{R}^n}|e^{\beta s}x|^{\mu p}|z(e^{\beta s}x)|^p\,dx =$$

$$e^{((n+\mu p)\beta-\alpha p)s}\int_{\mathbb{R}^n}|x|^{\mu p}(\mathbf{d}(s)z)(x)|^p\,dx<+\infty.$$

Since $e^{((n+\mu p)\beta-\alpha p)s}>0$ for any $\alpha,\beta,p,s\in\mathbb{R}$ then $\mathbf{d}(s)z\in L^p_\mu(\mathbb{R}^n,\mathbb{R}^m)$ for any $s\in\mathbb{R}$. Obviously, $\mathbf{d}(s)$ is a linear operator on L^p_μ, i.e. $\mathbf{d}(s)(\mu_1 z_1+\mu z_2)=\mu_1\mathbf{d}(s)z_1+\mu_2\mathbf{d}(s)z_2$, for any $\mu_1,\mu_2\in\mathbb{R}$ and $z_1,z_2\in L^p_\mu(\mathbb{R}^n,\mathbb{R}^m)$. Moreover, the latter identities imply that

$$\|\mathbf{d}(s)z\|_{p,\mu}=e^{(\alpha-(n/p+\mu)\beta)s}\|z\|_p, \quad \|z\|_{p,\mu}:=\left(\int_{\mathbb{R}^n}|x|^{\mu p}|z(x)|^p\,dx\right)^{1/p}.$$

Hence, the operator $\mathbf{d}(s):L^p(\mathbb{R}^n,\mathbb{R}^m)\to L^p(\mathbb{R}^n,\mathbb{R}^m)$ is bounded for any $s\in\mathbb{R}$.

Let $p=\infty$. If $z\in L^\infty_\mu(\mathbb{R}^n,\mathbb{R}^m)$ then

$$\text{ess sup}|z(x)| = \text{ess sup}(|e^{\beta s}x|^\mu|z(e^{\beta s}x)|)<+\infty$$

for any $\beta,s,\mu\in\mathbb{R}$ and $\|\mathbf{d}(s)z\|_\infty=e^{(\alpha-\beta\mu)s}\|z\|_\infty$ for any $s\in\mathbb{R}$. Therefore, $\mathbf{d}(s)$ is also a linear bounded operator on $z\in L^\infty(\mathbb{R}^n,\mathbb{R}^m)$.

Obviously, $(\mathbf{d}(s)\circ\mathbf{d}(-s))z=(\mathbf{d}(-s)\circ\mathbf{d}(s))z$ for any $z:\mathbb{R}^n\to\mathbb{R}^n$ and any $s\in\mathbb{R}$ and we derive $[\mathbf{d}(s)]^{-1}=\mathbf{d}(-s)$.

3 Global Existence of Strong Solutions of Navier-Stokes Equations

Below for shortness we omit $\mathbb{R}^3$ in the notations of $\int_{\mathbb{R}^3}$, L^p, C_0^∞ and C_c^∞ if the context is clear. Without loss of generality (see e.g. [4, page 4]) we also assume that $\nu = 1$, where ν is a viscosity coefficient.

Let us consider the weak form of the Navier-Stokes equations

$$\int_{\mathbb{R}^3} u(0)\cdot\xi(0)+\int_0^T\int_{\mathbb{R}^3} u\cdot(\partial_t\xi+\Delta\xi)+p\,\mathrm{div}\xi=\int_0^T\int_{\mathbb{R}^3} u\cdot(u\cdot\nabla)\xi,\ \forall\xi\in C_0^\infty([0,T)\times\mathbb{R}^3,\mathbb{R}^3) \tag{3}$$

with $u(t)\in V$ for $t\in(0,T)$, where V is a set of the so-called weakly divergence free velocity fields:

$$V := \left\{u\in L^2 :\ \int_{\mathbb{R}^3} u\cdot\nabla\phi = 0,\ \forall\phi\in C_0^\infty(\mathbb{R}^3,\mathbb{R})\right\}.$$

The classical idea of analysis is to prove existence and regularity of weak solutions and next to show that any weak solution is smooth. For Navier-Stokes equations this analysis has been initiated by Jean Leray in 1938 (see [2]). We use the recent review [4] of his results.

Definition 1 ([4], Definition 3.7) A pair (u,p) is said to be a strong solution of the Navier-Stokes equations on $[0,T)$ if $u(t)\in V$ for $t\in(0,T)$, $p\in L^1_{loc}((0,T)\times\mathbb{R}^3,\mathbb{R})$,

$$u\in C([0,T),L^2)\cap C((0,T),L^\infty),$$

$\|u(t)\|_\infty$ is bounded as $t\to 0^+$ and (3) is satisfied.

A possible way for expansion of regularity to a larger set of initial conditions is to use the dilation symmetry as explained in the introduction. Several symmetries for Navier-Stokes equations are known (see e.g. [1] and references therein). Below we use just one of them (see [1, formula (1.5)]).

Lemma 3 *If (u,p) is a strong solution of the Navier-Stokes equations with the initial data $u(0)=u_0\in V\cap L^\infty$ defined on $[0,T)$ then for any $s\in\mathbb{R}$ the pair (u_s,p_s) given by*

$$u_s(t,x)=e^s u(e^{2s}t,e^s x),\quad p_s(t,x)=e^{2s}p(e^{2s}t,e^s x),\quad t\ge 0, x\in\mathbb{R}^n, s\in\mathbb{R}.$$

is a strong solution of the Navier-Stokes equations defined on $[0,e^{-2s}T)$ with the initial value $u_s(0)=\mathbf{d}(s)u_0$.

Proof 1) Let $\mathbf{d}_1(s)$ be defined by the formula (2) with $\alpha = 1$, $\beta = 1$.

Since u is a strong solution then the function $t \to u(t)$ is continuous in L^2 and in L^∞. According to Lemma 2, $\mathbf{d}_1(s)$ is a linear bounded invertible operator on L^2 and on L^∞. Hence, for any fixed $s \in \mathbb{R}$ the function $t \to \mathbf{d}_1(s)u(t)$ is also continuous in L^2 and L^∞, consequently, $u_s \in C([0, e^{-2s}T), L^2)$ and $u_s \in C((0, e^{-2s}T), L^\infty)$.

Notice $\|\mathbf{d}_1(s)z\|_\infty = e^s\|z\|_\infty$ for any $z \in L^\infty$ and any $s \in \mathbb{R}$ (see Lemma 2). Since $\|u(t)\|_\infty$ is bounded as $t \to 0^+$ then $\|u_s(t)\|_\infty = e^s\|u(e^s t)\|_\infty$ is also bounded as $t \to 0^+$.

If $u_0 \in V$ then $\mathbf{d}_1(s)u_0 \in V$ for any $s \in \mathbb{R}$. Indeed,

$$\int u_0 \cdot \nabla\phi = 0, \qquad \forall \phi \in C_0^\infty$$

and using the change-of-variable theorem in the Lebesgue integral we derive

$$0 = \int_{\mathbb{R}^3} u_0(x)\cdot\nabla\phi(x)dx = e^{3s}\int_{\mathbb{R}^3} u_0(e^s x)\cdot(\nabla\phi)(e^s x)dx = e^{2s}\int_{\mathbb{R}^3}(\mathbf{d}_1(s)u_0)\cdot\nabla\tilde{\phi},$$

where $\tilde{\phi} = \mathbf{d}_1(s)\phi$. Since $\mathbf{d}_1(s)$ maps C_0^∞ onto C_0^∞ (see Lemma 1) then

$$0 = \int (\mathbf{d}_1(s)u_0) \cdot \nabla\tilde{\phi}, \qquad \forall \tilde{\phi} \in C_0^\infty,$$

i.e. $\mathbf{d}_1(s)u_0 \in V$ for any $s \in \mathbb{R}$. Hence, the inclusion $u(t) \in V$, $t \in [0, T)$ implies $u_s(t) \in V$, $t \in [0, e^{-2s}T)$.

2) Let $\mathbf{d}_2(s)$ be given by the formula (1) with $\alpha = 2$, $\beta = 1$ and $\gamma = 2$. Since $p \in L^1_{loc}((0,T) \times \mathbb{R}^n, \mathbb{R})$ then

$$\int_0^T \int_{\mathbb{R}^3} |p(t,x)\xi(t,x)|dxdt < +\infty, \quad \forall \xi \in C_c^\infty((0,T) \times \mathbb{R}^3, \mathbb{R})$$

hence using the change-of-variable theorem in the Lebesgue integral for the functions $t \to e^{2s}t$ and $x \to e^s x$ we derive

$$\int_0^T \int_{\mathbb{R}^3} |p(t,x)\xi(t,x)|dxdt = e^{5s}\int_0^{e^{-2s}T}\int_{\mathbb{R}^3} |p(e^{2s}t, e^s x)\xi(e^{2s}t, e^s x)|dxdt =$$

$$e^s \int_0^{e^{-2s}T}\int_{\mathbb{R}^3} |p_s \cdot \mathbf{d}_2(s)\xi| < +\infty, \quad \forall \xi \in C_c^\infty((0,T) \times \mathbb{R}^3, \mathbb{R}).$$

Since the operator $\mathbf{d}_2(s)$ maps $C_c^\infty((0,T)\times\mathbb{R}^3, \mathbb{R})$ onto $C_c^\infty((0, e^{-2s}T)\times\mathbb{R}^3, \mathbb{R})$ then

$$\int_0^{e^{-2s}T}\int_{\mathbb{R}^3} |p_s \cdot \tilde{\xi}| < +\infty, \quad \forall \tilde{\xi} \in C_c^\infty((0, e^{-2s}T) \times \mathbb{R}^3, \mathbb{R}).$$

Therefore, $p_s \in L^1_{loc}((0, e^{-2s}T) \times \mathbb{R}^3, \mathbb{R})$.

3) Let us show that (u_s, p_s) satisfies the equation (3). Since (u,p) is a solution defined on $[0,+\infty)$ then $\forall \xi \in C_0^\infty([0,T)\times\mathbb{R}^3,\mathbb{R}^3)$ we have

$$\int_{\mathbb{R}^3} u(0,x)\cdot\xi(0,x)dx+\int_0^T\int_{\mathbb{R}^3} u(t,x)\cdot(\partial_t\xi(t,x)+(\Delta\xi)(t,x))+p(t,x)(\mathrm{div}\xi)(t,x)dx\,dt =$$

$$\int_0^T\int_{\mathbb{R}^3} u(t,x)\cdot(u(t,x)\cdot\nabla)\xi(t,x)dx\,dt$$

Using the change-of-variable theorem in the Lebesgue integral for the functions $t\to e^{2s}t$ and $x\to e^s x$ we derive

$$e^{3s}\int_{\mathbb{R}^3} u(0,e^s x)\cdot\xi(0,e^s x)dx + e^{5s}\int_0^{e^{-2s}T}\int_{\mathbb{R}^3} u(e^{2s}t,e^s x)\cdot(\partial_t\xi(e^{2s}t,e^s x)+$$

$$(\Delta\xi)(e^{2s}t,e^s x)) + p(e^{2s}t,e^{2s}x)(\mathrm{div}\xi)(e^{2s}t,e^s x)dx\,dt =$$

$$e^{5s}\int_0^{e^{-2s}T}\int_{\mathbb{R}^3} u(e^{2s}t,e^s x)\cdot(u(e^{2s}t,e^s x)\cdot\nabla)\xi(e^{2s}t,e^s x)dx\,dt$$

or, equivalently,

$$e^{2s}\int_{\mathbb{R}^3} u_s(0,x)\cdot\xi(0,e^s x)dx+e^{2s}\int_0^{e^{-2s}T}\int_{\mathbb{R}^3} e^{2s}u_s(t,x)\cdot\big(\partial_t\xi(e^{2s}t,e^s x)+(\Delta\xi)(e^{2s}t,e^s x)\big)$$

$$+e^s p_s(t,x)(\mathrm{div}\xi)(e^{2s}t,e^s x)dx\,dt =$$

$$e^{3s}\int_0^{e^{-2s}T}\int_{\mathbb{R}^3} u_s(t,x)\cdot(u_s(t,x)\cdot\nabla)\xi(e^{2s}t,e^s x)dx\,dt$$

Let us denote $\xi_s(t,x)=\xi(e^{2st},e^s x)$. Hence,

$$(\Delta\xi_s)(t,x) = e^{2s}(\Delta\xi)(e^{2st},e^s x),$$
$$(\nabla\xi_s)(t,x) = e^{s}(\nabla\xi)(e^{2s}t,e^s x),$$
$$(\partial_t\xi_s)(t,x) = e^{2s}(\partial_t\xi)(e^{2s}t,e^s x),$$
$$(\mathrm{div}\xi_s)(t,x) = e^{s}(\mathrm{div}\xi)(e^{2s}t,e^s x),$$

and

$$e^{2s}\int_R^3 u_s(0)\cdot\xi_s(0)+e^{2s}\int_0^{e^{-2s}T}\int u_s\cdot(\partial_t\xi_s+\Delta\xi_s+p_s(\mathrm{div}\xi_s)=e^{2s}\int_0^{e^{-2s}T}\int_{\mathbb{R}^3} u_s\cdot(u_s\cdot\nabla)\xi_s,$$

where $\xi_s \in C_0^\infty([0, e^{-2s}T), \times\mathbb{R}^3, \mathbb{R}^3)$. Since for any $s > 0$ the operator $\mathbf{d}_3(s)$ defined as

$$(\mathbf{d}_3(s)\xi)(t, x) = \xi(e^{2s}t, e^s x), \quad t > 0, x \in \mathbb{R}^3$$

maps $C_0^\infty([0, T) \times \mathbb{R}^3, \mathbb{R}^3)$ onto $C_0^\infty([0, e^{-2s}T), \times\mathbb{R}^3, \mathbb{R}^3)$ (see Lemma 1), then

$$\int_{\mathbb{R}^3} u_s(0)\cdot\xi_s(0) + \int_0^{e^{-2s}T}\int_{\mathbb{R}^3} u_s(t,x)\cdot(\partial_t\xi_s + \Delta\xi_s + p_s(\mathrm{div}\xi_s) = \int_0^{e^{-2s}T}\int_{\mathbb{R}^3} u_s(t,x)\cdot(u_s(t,x)\cdot\nabla)\xi_s.$$

holds for all $\xi_s \in C_0^\infty([0, e^{-\mu_s s}T) \times \mathbb{R}^3, \mathbb{R}^3)$. Therefore, (u_s, p_s) is a strong solution of the Navier-Stokes equations on $[0, e^{-2s}T)$ and $u_s(0) = \mathbf{d}(s)u_0$.

The proven lemma implies the following result, which describes the cases when global-in-time existence of strong solutions for small initial data is equivalent to global-in-time existence of strong solutions for large initial data.

Corollary 1 *Let $q_1, q_2 \in [1, \infty]$ and $r_1, r_2, \mu_1, \mu_2 \in \mathbb{R}$ and*

$$r_1(1 - \mu_1 - 3/q_1) + r_2(1 - \mu_2 - 3/q_2) \neq 0.$$

A strong solution of the Navier-Stokes equations with arbitrary the initial data $u(0) = u_0 \in V \cap L^{q_1}_{\mu_1} \cap L^{q_2}_{\mu_2}$ exists on $[0, +\infty)$ if and only if there exist $\varepsilon > 0$ such that for any

$$u_0 \in \{u \in V \cap L^{q_1}_{\mu_1} \cap L^{q_2}_{\mu_2} : \|u\|^{r_1}_{q_1,\mu_1}\|u\|^{r_2}_{q_2,\mu_2} < \varepsilon\}$$

a strong solution (u, p) with the initial data $u(0) = u_0$ exists on $[0, +\infty)$.

Proof Let $\mathbf{d}_1$ be defined as in the proof of Lemma 3. Assume that for any $u_0 \in \{u \in V \cap L^{q_1}_{\mu_1} \cap L^{q_2}_{\mu_2} : \|u\|^{r_1}_{q_1,\mu_1}\|u\|^{r_2}_{q_2,\mu_2} < \varepsilon\}$ there exists a global in time strong solution with $u(0) = u_0$ and let us show that for any $u_0 \in V \cap L^{q_1}_{\mu_1} \cap L^{q_2}_{\mu_2}$: $\|u_0\|^{r_1}_{q_1,\mu_1}\|u_0\|^{r_2}_{q_2,\mu_2} \geq \varepsilon$ then the Navier-Stokes equations also have a strong solution on $[0, +\infty)$.

In the proof of Lemma 3 we have shown that $\mathbf{d}_1(s)u_0 \in V$ for any $s \in \mathbb{R}$, by Lemma 2 and $\mathbf{d}_1(s)u_0 \in L^{q_1}_{\mu_1}$ and for $\mathbf{d}_1(s)u_0 \in L^{q_2}_{\mu_1}$ any $s \in \mathbb{R}$. By Lemma 2 we also derive

$$\|\mathbf{d}_1(s)u_0\|_{q_1,\mu} = e^{s(1-\mu_1-3/q_1)}\|u_0\|_{q_1,\mu_1},$$

and

$$\|\mathbf{d}_1(s)u_0\|_{q_2,\mu} = e^{s(1-\mu_2-3/q_2)}\|u_0\|_{q_2,\mu_2}$$

and

$$\|\mathbf{d}_1(s)u_0\|^{r_1}_{q_1,\mu_1}\|\mathbf{d}_1(s)u_0\|^{r_1}_{q_2,\mu_2} = e^{s(r_1(1-\mu_1-\frac{3}{q_1})+r_2(1-\mu_2-\frac{3}{q_2}))}\|u_0\|^{r_1}_{q_1,\mu_1}\|u_0\|^{r_2}_{q_2,\mu_2}.$$

Since by assumption $r_1(1-\mu_1-3/q_1)+r_2(1-\mu_2-3/q_2)\neq 0$ then for any $u_0\in V\cap L^{q_1}_{\mu_1}\cap L^{q_2}_{\mu_2}$ there exists $s_0\in\mathbb{R}$ such that

$$\|\mathbf{d}_1(s)u_0\|_{q_1}^{r_1}\|\mathbf{d}_1(s)u_0\|_{q_2}^{r_2}<\varepsilon.$$

Hence, if a strong solution (u,p) with $u(0)=\mathbf{d}_1(s_0)u_0$ exists on $[0,+\infty)$ then by Lemma 3 the pair $(\tilde{u},\tilde{p})$ given by

$$\tilde{u}(t,x)=e^{-s_0}u(e^{-2s_0}t,e^{-s_0}x),\tilde{p}(t,x)=e^{-2s_0}p(e^{-2s_0}t,e^{-s_0}x),t\in[0,+\infty),x\in\mathbb{R}^3$$

is also a strong solution of the Navier-Stokes equation. Since $u(0)=\mathbf{d}_1(s_0)u_0$ means that

$$u(0,x)=e^{s_0}u_0(e^{s_0}x),\quad x\in\mathbb{R}^3$$

then

$$\tilde{u}(0,x)=e^{-s_0}u(0,e^{-s_0}x)=u_0(x),$$

i.e. $\tilde{u}(0)=u_0\in V\cap L^{q_1}_{\mu_1}\cap L^{q_2}_{\mu_2}$ and the proof is complete.

Notice that taking $\mu_1=\mu_2=0$ we derive the usual L^{q_1} and L^{q_2} spaces in the latter corollary.

Let us mention the following properties of the strong solutions (see Definition 1) proven before:

- *Global-in-time existence for small initial data* [4, Corollary 3.13 and Lemma 3.10]
  There exist $\varepsilon>0$ and $C>0$ such that for any
  $$u_0\in\{u\in V\cap L^\infty\cap L^2:\|u\|_2^2\|u\|_\infty<\varepsilon\}\tag{4}$$
  or
  $$u_0\in\{u\in V\cap L^\infty\cap L^2:\|u\|_2\|\nabla u\|_2<\varepsilon\}\tag{5}$$
  or
  $$u_0\in\{u\in V\cap L^\infty\cap L^2:\|u\|_2^{2(q-3)}\|u\|_q^q<\varepsilon\},q>3\tag{6}$$
  a strong solution (u,p) with the initial data $u(0)=u_0$ exists on $[0,+\infty)$ and $\|u(t)\|_\infty\leq C\|u_0\|_\infty$.
- *Uniqueness of strong solutions* [4, Theorem 3.9]
  For any $u_0\in V\cap L^\infty$ a strong solution with $u(0)=u_0$ is unique.
- *Smothness* [4, Corollary 3.3].
  If (u,p) is a strong solution of the Navier-Stokes equation then
  $$\partial_t^k\nabla^m u,\partial_t^k\nabla^m p\in C((0,T),L^2)\cap C((0,T),L^\infty),\quad\forall m,k\geq 0$$
  and, in particular, $u,p\in C^\infty(\mathbb{R}^3\times(0,T))$ constitute a classical solution of the Navier-Stokes equations on $(0,T)\times\mathbb{R}^3$.

None of conditions (4), (5), (6) satisfy Corollary 1.

To expand globally the regularity of the Navier-Stokes equation a global-in-time existence of strong solutions for small initial data has to be proven for the norms satisfying Corollary 1.

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